Application of the Campbell-Magaard theorem
to higher-dimensional physics

Sanjeev S. Seahra[*] and Paul S. Wesson[†]
Department of Physics, University of Waterloo
Waterloo, Ontario, N2L 3G1, Canada
(Dated: February 5, 2003)

Stated succinctly, the original version of the Campbell-Magaard theorem says that it is always possible to locally embed any solution of 4-dimensional general relativity in a 5-dimensional Ricci-flat manifold. We discuss the proof of this theorem (and its variants) in n dimensions, and its application to current theories that postulate that our universe is a 4-dimensional hypersurface Σ_0 within a 5-dimensional manifold, such as Space-Time-Matter (STM) theory and the Randall & Sundrum (RS) braneworld scenario. In particular, we determine whether or not arbitrary spacetimes may be embedded in such theories, and demonstrate how these seemingly disparate models are interconnected. Special attention is given to the motion of test observers in 5 dimensions, and the circumstances under which they are confined to Σ_0. For each 5-dimensional scenario considered, the requirement that observers be confined to the embedded spacetime places restrictions on the 4-geometry. For example, we find that observers in the thin braneworld scenario can be localized around the brane if its total stress-energy tensor obeys the 5-dimensional strong energy condition. As a concrete example of some of our technical results, we discuss a $\mathbb{Z}_2$ symmetric embedding of the standard radiation-dominated cosmology in a 5-dimensional vacuum.

PACS numbers: 04.50.+h, 04.20.Cv

I. INTRODUCTION

The idea that our universe might be a 4-dimensional hypersurface embedded in a higher-dimensional manifold is an old one with a long history, as well as the subject of a considerable amount of contemporary interest. The primordial impetus for this line of study comes from the work of Kaluza [1], who showed that one can obtain a classical unification of gravity and electromagnetism by adding an extra dimension to spacetime (1921); and Klein [2], who suggested that extra dimensions have a circular topology of very small radius and are hence unobservable (1926). The latter idea, the so-called "compactification" paradigm, came to dominate most approaches to higher-dimensional physics, the most notable of which was early superstring theory. However, a number of papers have appeared in the intervening years that do not assume extra dimensions with compact topologies; early examples include the works of Joseph [3], Akama [4], Rubakov & Shaposhnikov [5], Visser [6], Gibbons & Wiltshire [7], and Antoniadis [8]. A systematic and independent approach to the 5-dimensional, noncompact Kaluza-Klein scenario, known as Space-Time-Matter (STM) theory, followed [9, 10, 11, 12]. Then, in 1996 Horava & Witten showed that the compactification paradigm was not a prerequisite of string theory with their discovery of an 11-dimensional theory on the orbifold $\mathbb{R}^{10} \times S^1/\mathbb{Z}_2$, which is related to the 10-dimensional $E_8 \times E_8$ heterotic string via dualities [13]. In this theory,

standard model interactions are confined to a 3-brane, on which the endpoints of open strings reside, while gravitation propagates in the higher-dimensional bulk. This situation has come to be known as the "braneworld scenario." The works of Arkani-Hamed *et al.* [14, 15, 16] and Randall & Sundrum (RS) [17, 18], which used noncompact extra dimensions to address the hierarchy problem of particle physics and demonstrated that the graviton ground state can be localized on a 3-brane in 5 dimensions, won a large following for the braneworld scenario. A virtual flood of papers dealing with noncompact, higher-dimensional models of the universe soon followed, including works dealing with "thick" branes — that is, 3-branes with finite extra-dimensional width [19, 20].

On the other hand, the abstract problem of embedding an n-dimensional (pseudo-) Riemannian manifold Σ_0 in a higher-dimensional space M also has a rich pedigree[1]. Soon after Riemann published the theory of intrinsically-defined abstract manifolds in 1868 [22], Schläfli considered the problem of how to locally embed such manifolds in Euclidean space [23]. He conjectured that the maximum number of extra dimensions necessary for a local embedding was $\frac{1}{2}n(n-1)$. In 1926, Janet provided a partial proof of the conjecture for $n = 2$ using power series methods [24]. That result was soon generalized to arbitrary n by Cartan [25]. The proof was completed by Burstin [26], who demonstrated that the Gauss-Codazzi-Ricci equations were the integrability conditions of the embedding. A related embedding problem was first con-

---

[1] An extensive bibliography is available from ref. [21].

sidered by Campbell in 1926 [27]: How many extra dimensions are required to locally embed Σ_0 in a Ricci-flat space; i.e., a higher-dimensional vacuum spacetime? He proposed that the answer was one, which was later proved by Magaard [28]. The Campbell-Magaard theorem has recently been extended to include cases where the higher-dimensional space has a nonzero cosmological constant [29, 30], is sourced by a scalar field [31], and has an arbitrary non-degenerate Ricci tensor [32]. It is important to note that the results discussed thus far are local; the problem of global embedding arbitrary submanifolds is more difficult. In 1956, Nash showed that the minimum number of extra dimension required to embed Σ_0 in Euclidean space is $3n(n+3)/2$ if Σ_0 is compact — it increases to $3n(n+1)(n+3)/2$ if Σ_0 is non-compact [33]. Global results for metrics with indefinite signature were later obtained by Clarke [34] and Greene [35].

It is obvious that there is a significant interplay between higher-dimensional theories of the universe and mathematical embedding theorems. Specifically, the Campbell-Magaard theorem and its variants would seem to be of particular importance to higher-dimensional physics because the embedding space in such theories is usually specified solely by its stress-energy — and hence Ricci — tensor. Yet this theorem has received only moderate coverage in the literature [36, 37, 38, 39]. One of the primary motivations of this paper is to give an introduction to, and a non-rigourous proof of, the Campbell-Magaard theorem (Sec. II). In the process, we will see how the theorem can be altered to fit any number of circumstances. For example, we will see that it is possible to successfully embed Σ_0 in M with arbitrary extrinsic (as opposed to intrinsic) curvature. We will then discuss general geometric properties of several 5-dimensional theories — STM and the thin/thick braneworld scenarios — as well as the application of the Campbell-Magaard theorem to each (Sec. III). Our discussion will demonstrate that these seemingly disparate models actually have a lot in common. We will also generalize the $(4+1)$-splitting of the 5-dimensional geodesic equation found in ref. [40] to analyze the motion of test observers in each scenario. We will see that in all the cases considered, the physical demand that observer trajectories are confined to Σ_0 makes it impossible to successfully implement a 5-dimensional embedding of arbitrary 4-manifolds. However, the restrictions placed on the geometry of embedded spacetimes by this requirement are different for each scenario. Finally, we will illustrate several aspects of the proceeding discussion by considering the $\mathbb{Z}_2$ symmetric embedding of standard radiation-dominated cosmologies in a 5-dimensional vacuum manifold. This embedding is obtained from a known solution of STM theory found in ref. [41]. We will see that observers can be confined to Σ_0 in this case, but that their equilibrium is unstable. Sec. V is reserved for a summary of our results.

Conventions. Uppercase Latin indices run from $0 \ldots n$, lowercase Greek indices run $0 \ldots (n-1)$, and lowercase Latin indices run $1 \ldots \frac{3}{2}n(n+1)$. Square brackets on indices indicate anti-symmetrization. The $(n+1)$-dimensional covariant derivative will be indicated by ∇_A, while the n-dimensional covariant derivative will be indicated with a semicolon. Curvature tensors in $(n+1)$ dimensions are distinguished from their n-dimensional counterparts with a hat; i.e. $\hat{R}_{AB}$ versus $R_{\alpha\beta}$. We choose units where $c = 1$. The coupling constant between the n-dimensional Einstein and stress-energy tensors is κ_n^2 such that $G_{\alpha\beta} = \kappa_n^2 T_{\alpha\beta}$.

II. THE GENERAL EMBEDDING PROBLEM

In this section, we discuss the general problem of embedding an n-dimensional spacetime in an $(n+1)$-dimensional manifold. Certain aspects of our discussion may be familiar to some readers from other contexts. Our philosophy will to be to ignore such previous knowledge and start from basic geometric principles. In this way, we hope to emphasize various points that play an important role in the applications discussed in Sec. III.

A. Geometric Construction

We will be primarily concerned with an $(n+1)$-dimensional manifold (M, g_{AB}) on which we place a coordinate system $x \equiv \{x^A\}$. In our working, we will allow for two possibilities: either there is one timelike and n spacelike directions tangent to M, *or* there are two timelike and $(n-1)$ spacelike directions tangent to M. We introduce a scalar function

$$\ell = \ell(x), \tag{1}$$

which defines our foliation of the higher-dimensional manifold with the hypersurfaces given by $\ell = \text{constant}$, denoted by Σ_ℓ. If there is only one timelike direction tangent to M, we assume that the vector field n^A normal to Σ_ℓ is spacelike. If there are two timelike directions, we take the unit normal to be timelike. In either case, the space tangent to a given Σ_ℓ hypersurface contains one timelike and $(n-1)$ spacelike directions. That is, each Σ_ℓ hypersurface corresponds to an n-dimensional Lorentzian spacetime. The normal vector to the Σ_ℓ slicing is given by

$$n_A = \varepsilon \Phi \, \partial_A \ell, \quad n^A n_A = \varepsilon. \tag{2}$$

Here, $\varepsilon = \pm 1$. The scalar Φ which normalizes n^A is known as the lapse function. We define the projection tensor as

$$h_{AB} = g_{AB} - \varepsilon n_A n_B. \tag{3}$$

This tensor is symmetric ($h_{AB} = h_{BA}$) and orthogonal to n_A. We place an n-dimensional coordinate system on each of the Σ_ℓ hypersurfaces $y \equiv \{y^\alpha\}$. The n basis vectors

$$e_\alpha^A = \frac{\partial x^A}{\partial y^\alpha}, \quad n_A e_\alpha^A = 0 \tag{4}$$

are by definition tangent to the Σ_ℓ hypersurfaces and orthogonal to n^A. It is easy to see that e_α^A behaves as a vector under coordinate transformations on M [$\phi : x \to \bar{x}(x)$] and a one-form under coordinate transformations on Σ_ℓ [$\psi : y \to \bar{y}(y)$]. We can use these basis vectors to project higher-dimensional objects onto Σ_ℓ hypersurfaces. For example, for an arbitrary one-form on M we have

$$T_\alpha = e_\alpha^A T_A. \tag{5}$$

Here T_α is said to be the projection of T_A onto Σ_ℓ. Clearly T_α behaves as a scalar under ϕ and a one-form under ψ. The induced metric on the Σ_ℓ hypersurfaces is given by

$$h_{\alpha\beta} = e_\alpha^A e_\beta^B g_{AB} = e_\alpha^A e_\beta^B h_{AB}. \tag{6}$$

Just like g_{AB}, the induced metric has an inverse:

$$h^{\alpha\gamma} h_{\gamma\beta} = \delta_\beta^\alpha. \tag{7}$$

The induced metric and its inverse can be used to raise and lower the indices of tensors tangent to Σ_ℓ, and change the position of the spacetime index of the e_α^A basis vectors. This implies

$$e_A^\alpha e_\beta^A = \delta_\beta^\alpha. \tag{8}$$

Also note that since h_{AB} is entirely orthogonal to n^A, we can express it as

$$h_{AB} = h_{\alpha\beta} e_A^\alpha e_B^\beta. \tag{9}$$

At this juncture, it is convenient to introduce our definition of the extrinsic curvature $K_{\alpha\beta}$ of the Σ_ℓ hypersurfaces:

$$K_{\alpha\beta} = e_\alpha^A e_\beta^B \nabla_A n_B = \tfrac{1}{2} e_\alpha^A e_\beta^B \pounds_n h_{AB}. \tag{10}$$

Note that the extrinsic curvature is symmetric ($K_{\alpha\beta} = K_{\beta\alpha}$). It may be thought of as the derivative of the induced metric in the normal direction. This n–tensor will appear often in what follows.

Finally, we note that $\{y, \ell\}$ defines an alternative coordinate system to x on M. The appropriate diffeomorphism is

$$dx^A = e_\alpha^A dy^\alpha + \ell^A d\ell, \tag{11}$$

where

$$\ell^A = \left(\frac{\partial x^A}{\partial \ell} \right)_{y^\alpha = \text{const.}} \tag{12}$$

is the vector tangent to lines of constant y^α. We can always decompose 5D vectors into the sum of a part tangent to Σ_ℓ and a part normal to Σ_ℓ. For ℓ^A we write

$$\ell^A = N^\alpha e_\alpha^A + \Phi n^A. \tag{13}$$

This is consistent with $\ell^A \partial_A \ell = 1$, which is required by the definition of ℓ^A, and the definition of n^A. The n–vector N^α is the shift vector, which describes how the y^α coordinate system changes as one moves from a given Σ_ℓ hypersurface to another. Using our formulae for dx^A and ℓ^A, we can write the 5D line element as

$$\begin{aligned} ds_{(5)}^2 &= g_{AB} \, dx^A dx^B \\ &= h_{\alpha\beta}(dy^\alpha + N^\alpha d\ell)(dy^\beta + N^\beta d\ell) \\ &\quad + \varepsilon \Phi^2 d\ell^2. \end{aligned} \tag{14}$$

This reduces to $d\mathcal{S}^2 = h_{\alpha\beta} dy^\alpha dy^\beta$ if $d\ell = 0$, a case of considerable physical interest.

B. Decomposition of the higher-dimensional field equations

In this section, we describe how n-dimensional field equations on each of the Σ_ℓ hypersurfaces can be derived, given that the $(n+1)$-dimensional field equations are

$$\hat{R}_{AB} = \lambda g_{AB}, \quad \lambda \equiv \frac{2\Lambda}{1-n}. \tag{15}$$

Here Λ is the "bulk" cosmological constant, which may be set to zero if desired. In what follows, we will extent the 4-dimensional usage and call manifolds satisfying equation (15) Einstein spaces.

Our starting point is the Gauss-Codazzi equations. On each of the Σ_ℓ hypersurfaces these read

$$\hat{R}_{ABCD} e_\alpha^A e_\beta^B e_\gamma^C e_\delta^D = R_{\alpha\beta\gamma\delta} + 2\varepsilon K_{\alpha[\delta} K_{\gamma]\beta}, \tag{16a}$$
$$\hat{R}_{MABC} n^M e_\alpha^A e_\beta^B e_\gamma^C = 2K_{\alpha[\beta;\gamma]}. \tag{16b}$$

These need to be combined with the following expression for the higher-dimensional Ricci tensor:

$$\hat{R}_{AB} = (h^{\mu\nu} e_\mu^M e_\nu^N + \varepsilon n^M n^N) \hat{R}_{AMBN}. \tag{17}$$

The $\tfrac{1}{2}(n+1)(n+2)$ separate equations for the components of $\hat{R}_{AB}$ may be broken up into three sets by considering the following contractions of equation (15):

$$\begin{array}{ll} \hat{R}_{AB} e_\alpha^A e_\beta^B = \lambda h_{\alpha\beta}, & \tfrac{1}{2}n(n+1) \text{ equations,} \\ \hat{R}_{AB} e_\alpha^A n^B = 0, & n \text{ equations,} \\ \hat{R}_{AB} n^A n^B = \varepsilon\lambda, & \underline{+1 \text{ equation,}} \\ & \tfrac{1}{2}(n+1)(n+2) \text{ equations.} \end{array} \tag{18}$$

Putting (17) into (18) and making use of (16) yields the following formulae:

$$R_{\alpha\beta} = \lambda h_{\alpha\beta} - \varepsilon[E_{\alpha\beta} + K_\alpha{}^\mu(K_{\beta\mu} - Kh_{\beta\mu})], \tag{19a}$$
$$0 = (K^{\alpha\beta} - h^{\alpha\beta}K)_{;\alpha}, \tag{19b}$$
$$\varepsilon\lambda = E_{\mu\nu} h^{\mu\nu}. \tag{19c}$$

In writing down these results, we have made the following definitions:

$$K \equiv h^{\alpha\beta} K_{\alpha\beta}, \tag{20}$$

$$E_{\alpha\beta} \equiv \hat{R}_{MANB} n^M e^A_\alpha n^N e^B_\beta, \quad E_{\alpha\beta} = E_{\beta\alpha}. \qquad (21)$$

The Einstein tensor $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$ on a given Σ_ℓ hypersurface is given by

$$
\begin{aligned}
G^{\alpha\beta} =& -\varepsilon \left(E^{\alpha\beta} + K^\alpha{}_\mu P^{\mu\beta} - \tfrac{1}{2} h^{\alpha\beta} K^{\mu\nu} P_{\mu\nu} \right) \\
& -\tfrac{1}{2}\lambda(n-3) h^{\alpha\beta},
\end{aligned}
\qquad (22)
$$

where we have defined the (conserved) tensor

$$P_{\alpha\beta} \equiv K_{\alpha\beta} - h_{\alpha\beta} K, \quad P^{\alpha\beta}{}_{;\beta} = 0. \qquad (23)$$

Some will recognize this as algebraically equivalent the momentum conjugate to the induced metric in the ADM Hamiltonian formulation of general relativity.[2] The condition that the Einstein 4-tensor has zero divergence translates into a condition satisfied by $E_{\alpha\beta}$:

$$E^{\alpha\beta}{}_{;\beta} = \varepsilon(K_{\mu\nu} K^{\mu\nu;\alpha} - K^{\mu\beta} K^\alpha{}_{\mu;\beta}). \qquad (24)$$

In deriving this formula, we have made use of (19b). Equations (19) are the field equations governing n-dimensional curvature quantities on the Σ_ℓ hypersurfaces. We will study the properties of these field equations in the next section.

Before moving on, we note that it is possible to solve equation (19a) for $E_{\alpha\beta}$ and substitute the result into (19c) to get

$$(n-1)\lambda = R + \varepsilon(K^{\mu\nu} K_{\mu\nu} - K^2). \qquad (25)$$

Taken together, equations (19b) and (25) are $(n+1)$-dimensional generalizations of the well-known Hamiltonian constraints, familiar to those who work with numerical relativity or initial-value problems in $(3+1)$ dimensions. So these formulae have been written down before, and similar equations have been used to analyze the RS scenario by Shiromizu *et al.* [42], though not in the context of the theorem we are about to describe.

C. The generalized Campbell-Magaard theorem

In this section, we discuss generic features of the embedding problem and outline a proof of the Campbell-Magaard theorem based on the formulae derived above. Technical comments can be found in refs. [28, 29, 30, 31, 32], but here we wish to provide a physically-motivated argument.

In the previous section, we derived field equations for the n-dimensional tensors — which can be thought of as

---

three spin-2 fields — defined on each of the Σ_ℓ hypersurfaces:

$$h_{\alpha\beta}(y,\ell), \quad K_{\alpha\beta}(y,\ell), \quad E_{\alpha\beta}(y,\ell). \qquad (26)$$

Each of the tensors is symmetric, so there are $3 \times \frac{1}{2} n(n+1)$ independent dynamical quantities governed by the field equations (19). For book-keeping purposes, we can organize these into an $n_{\rm dyn} = \frac{3}{2} n(n+1)$-dimensional supervector $\Psi^a = \Psi^a(y,\ell)$. Now, the field equations (19) contain no derivatives of the tensors (26) with respect to ℓ. This means that the components $\Psi^a(y,\ell)$ must satisfy (19) and (24) for *each and every* value of ℓ. In an alternative language, the field equations on Σ_ℓ are "conserved" as we move from hypersurface to hypersurface. That is, the field equations (19) in $(n+1)$-dimensions are, in the Hamiltonian sense, *constraint equations*. While this is important from the formal viewpoint, in means that equations (15) tell us nothing about how Ψ^a varies with ℓ. Equations governing the ℓ–evolution of Ψ^a may be derived in a number of equivalent ways. These include isolating ℓ-derivatives in the expansion of the Bianchi identities $\nabla_A G^{AB} = 0$; direct construction of the Lie derivatives of h_{AB} and $K_{AB} = h_A{}^C \nabla_C n_B$ with respect to ℓ^A; and formally re-expressing the gravitational Lagrangian as a Hamiltonian (with ℓ playing the role of time) to obtain the equations of motion. Because the derivation of $\partial_\ell \Psi^a$ is tedious and not really germane to our discussion, we will omit it from our considerations and turn to the more important problem of embedding.

Essentially, our goal is to find a solution of the higher-dimensional field equations (15) such that *one* hypersurface Σ_0 in the Σ_ℓ foliation has "desirable" geometrical properties. For example, we may want to completely specify the induced metric on, and hence the intrinsic geometry of, Σ_0. Without loss of generality, we can assume that the hypersurface of interest is at $\ell = 0$. Then to successfully embed Σ_0 in M, we need to do two things:

1. Solve the constraint equations (19) on Σ_0 for $\Psi^a(y,0)$ such that Σ_0 has the desired properties (this involves physics).

2. Obtain the solution for $\Psi^a(y,\ell)$ in the bulk (i.e. for $\ell \neq 0$) using the evolution equations $\partial_\ell \Psi^a$ (this is mainly mathematics).

To prove the Campbell-Magaard theorem one has to show that Step 1 is possible for arbitrary choices of $h_{\alpha\beta}$ on Σ_0, and one also needs to show that the bulk solution for Ψ^a obtained in Step 2 preserves the equations of constraint on $\Sigma_\ell \neq \Sigma_0$. The latter requirement is necessary because if the constraints are not conserved, the higher-dimensional field equations will not hold away from Σ_0. This issue has been considered by several authors, who have derived evolution equations for Ψ^a and demonstrated that the constraints are conserved in quite general $(n+1)$-dimensional manifolds [29, 30, 32]. Rather than dwell on this well-understood point, we will concentrate on the n-dimensional field equations on Σ_0 and

assume that, given $\Psi^a(y,0)$, then the rest of $(n+1)$-dimensional geometry can be generated using evolution equations, with the resulting higher-dimensional metric satisfying the appropriate field equations. However, we expect that the practical implementation of the formal embedding procedure given above will be fraught with the same type of computational difficulties associated with the initial-value problem in ordinary general relativity, and is hence a nontrivial exercise.

Now, there are $n_{\rm cons} = \frac{1}{2}(n+1)(n+2)$ constraint equations on Σ_0. For $n \geq 2$ we see that $\dim \Psi^a = n_{\rm dyn}$ is greater than $n_{\rm cons}$, which means that our system is underdetermined. Therefore, we may freely specify the functional dependence of $n_{\rm free} = n^2 - 1$ components of $\Psi^a(y,0)$. This freedom is at the heart of the Campbell-Magaard theorem. Since $n_{\rm free}$ is greater than the number of independent components of $h_{\alpha\beta}$ for $n \geq 2$, we can choose the line element on Σ_0 to correspond to any n-dimensional Lorentzian manifold and still satisfy the constraint equations. This completes the proof of the theorem, *any n-dimensional manifold can be locally embedded in an $(n+1)$-dimensional Einstein space.*

We make a few comments before moving on: First, it is equally valid to fix $K_{\alpha\beta}(y,0)$ — or even $E_{\alpha\beta}(y,0)$ — instead of $h_{\alpha\beta}(y,0)$. That is, instead of specifying the *intrinsic* curvature of Σ_0, one could arbitrarily choose the *extrinsic* curvature. However, it is obvious that we cannot arbitrarily specify *both* the induced metric and extrinsic curvature of Σ_0, *and* still solve the constraints: there are simply not enough degrees of freedom. We will see in Sec. III that there are several scenarios where we will want to set $K_{\alpha\beta}(y,0) = 0$, but that in doing so we will severely restrict the intrinsic geometry of Σ_0.

Second, one might legitimately wonder about the $n_{\rm residual} = n_{\rm dyn} - n_{\rm cons} - \frac{1}{2}n(n+1) = \frac{1}{2}(n+1)(n-2)$ degrees of freedom in $\Psi^a(y,0)$ "left over" after the constraints are imposed and the induced metric is selected. What role do these play in the embedding? The existence of some degree of arbitrariness in $\Psi^a(y,0)$, which essentially comprises the initial data for a Cauchy problem, would seem to suggest that when we choose an induced metric on Σ_0 we do not uniquely fix the properties of the bulk. In other words, we can apparently embed the same n-manifold in different Einstein spaces. For example, in Sec. IV we will see how a 4-dimensional radiation-dominated cosmological model can be embedded in a 5-dimensional bulk with $\hat{R}_{AB} = 0$ and $\hat{R}_{ABCD} \neq 0$. However, in refs. [43, 44] it was demonstrated that the same 4-manifold can been embedded in 5-dimensional Minkowski space. These results confirm that, in general, the structure of M is not determined uniquely by the intrinsic geometry of Σ_0.

Third, we reiterate that the Campbell-Magaard theorem is a local result. It does not state that it is possible to embed Σ_0 with arbitrary global topology into an $(n+1)$-dimensional Einstein space. As far as we know, the issue of how many extra dimensions are required for a global embedding of Σ_0 an Einstein space is an open question.

III. APPLICATIONS

We now want to apply the formalism discussed in Sec. II to three higher-dimensional theories of our universe: STM theory, the RS thin braneworld scenario, and the thick braneworld scenario. The common feature of these models is that our $(3+1)$-dimensional spacetime is viewed as a hypersurface in a 5-dimensional manifold. For obvious reasons, we take $n = 4$ when using equations from Sec. II in the current discussion. Then, each Σ_ℓ hypersurface can be identified with a 4-dimensional embedded spacetime.

A. Space-Time-Matter theory

STM theory predates the current flurry of interest in non-compact 5-dimensional models spurred by the work of RS. Essentially, it is a *minimal* model that assumes the 5-dimensional manifold to be devoid of matter. That is, the 5-manifold is Ricci-flat. The field equations of this theory are then given by equations (19) with $\lambda = 0$. Despite the fact that we have $\hat{G}_{AB} = 0$, the Einstein tensor on each of the embedded spacetimes is non-trivial and is given by equation (22) with vanishing bulk cosmological constant:

$$G^{\alpha\beta} = -\varepsilon \left(E^{\alpha\beta} + K^\alpha{}_\mu P^{\mu\beta} - \tfrac{1}{2}h^{\alpha\beta}K^{\mu\nu}P_{\mu\nu} \right). \quad (27)$$

Matter enters into STM theory when we consider an observer who is capable of performing experiments that measure the 4-metric $h_{\alpha\beta}$ or Einstein tensor $G_{\alpha\beta}$ in some neighbourhood of their position, yet is ignorant of the dimension transverse to his spacetime, the 5-metric g_{AB} and 5-dimensional curvature tensors. For general situations, such an observer will discover that his universe is curved, and that the local Einstein tensor is given by (27). Now, if this person believes in the Einstein equations $G_{\alpha\beta} = \kappa_4^2 T_{\alpha\beta}$, he will be forced to conclude that the spacetime around him is filled with some type of matter field. This is a somewhat radical departure from the usual point of view that the stress-energy distribution of matter fields acts as the source of the curvature of the universe. In the STM picture, the shape of the Σ_ℓ hypersurfaces plus the 5-dimensional Ricci-flat geometry fixes the matter distribution. It is for this reason that STM theory is sometimes called induced-matter theory: the matter content of the universe is induced from higher-dimensional geometry. This geometrization of matter is the primary motivation for studying STM theory. (For an in-depth review of this formalism, see ref. [12].)

When applied to STM theory, the Campbell-Magaard theorem says that it is possible to specify the form of $h_{\alpha\beta}$ on one of the embedded spacetimes, denoted by Σ_0. In other words, *we can take any known $(3+1)$-dimensional solution $h_{\alpha\beta}$ of the Einstein equations for matter with stress-energy tensor $T_{\alpha\beta}$ and embed it on a hypersurface in the STM scenario.* The stress-energy tensor of the

induced matter on that hypersurface Σ_0 will necessarily match that of the $(3+1)$-dimensional solution. However, there is no guarantee that the induced matter on any of the other spacetimes will have the same properties.

We now wish to expand the discussion to include the issue of observer trajectories in STM theory. To do this, we will need the covariant decomposition of the equation of motion for test observers into parts describing tangential and orthogonal accelerations; this is accomplished in Appendix A. The salient result from that discussion is equation (A6), which governs motion in the ℓ direction. We now analyze this formula for three different cases that may apply to a given Σ_ℓ 4-surface in an STM manifold. (The various quantities appearing in the following are defined in Appendix A.)

1. $K_{\alpha\beta} \neq 0$ *and* $\mathfrak{F} = 0$. A sub-class of this case has $f^\beta = 0$, which corresponds to freely-falling observers. We cannot have $\ell = $ constant as a solution of the ℓ equation of motion (A6) in this case, so observers cannot live on a single hypersurface. Therefore, if we construct a Ricci-flat 5-dimensional manifold in which a particular solution of general relativity is embedded on Σ_0, and we put an observer on that hypersurface, then that observer will inevitability move in the ℓ direction. Therefore, the properties of the induced matter that the observer measures may match the predictions of general relativity for a brief period of time, but this will not be true in the long run. Therefore, STM theory predicts observable departures from general relativity.

2. $K_{\alpha\beta} = 0$ *and* $\mathfrak{F} = 0$. Again, this case includes freely-falling observers. Here, we can solve equation (A6) with $d\ell/d\lambda = 0$. That is, if a particular hypersurface Σ_0 has vanishing extrinsic curvature, then we can have observers with trajectories contained entirely within that spacetime, provided that $\mathfrak{F} = 0$. Such hypersurfaces are called geodesically complete because every geodesic on Σ_0 is also a geodesic of M [40, 45]. If we put $K_{\alpha\beta} = 0$ into equation (27), then we get the Einstein tensor on Σ_ℓ as

$$G_{\alpha\beta} = \kappa_4^2 T_{\alpha\beta} = -\varepsilon E_{\alpha\beta} \Rightarrow T^\alpha{}_\alpha = 0. \qquad (28)$$

This says that the stress-energy tensor of the induced matter associated with geodesically complete spacetimes must have zero trace. Assuming that the stress-energy tensor may expressed as that of a perfect fluid, this implies a radiation-like equation of state. Hence, it is impossible to embed an *arbitrary* spacetime in a 5-dimensional vacuum such that it is geodesically complete. This is not surprising, since we have already seen in Sec. II C that we cannot use the Campbell-Magaard theorem to choose both $h_{\alpha\beta}$ and $K_{\alpha\beta}$ on Σ_0 — we have the freedom to specify one or the other, but not both. If we do demand that test observers are gravitation-

ally confined to Σ_0, we place strong restrictions on the geometry of the embedded spacetime.

3. $K_{\alpha\beta} \neq 0$ *and* $\mathfrak{F} = -K_{\alpha\beta}u^\alpha u^\beta$. In this case, we can solve (A6) with $d\ell/d\lambda = 0$ and hence have observers confined to the Σ_0 spacetime. It was shown in ref. [40] that $\mathfrak{F} = -K_{\alpha\beta}u^\alpha u^\beta$ is merely the higher-dimensional generalization of the centripetal acceleration familiar from elementary mechanics. Since we do not demand $K_{\alpha\beta} = 0$ in this case, we can apply the Campbell-Magaard theorem and have any type of induced matter on Σ_0. However, the price to be paid for this is the inclusion of a non-gravitational centripetal confining force, whose origin is obscure.

To summarize, we have shown that the Campbell-Magaard theorem guarantees that we can embed any solution of general relativity on a spacetime hypersurface Σ_0 within the 5D manifold postulated by STM theory. However, in general situations observer trajectories will not be confined to Σ_0. The exception to this is when Σ_0 has $K_{\alpha\beta} = 0$: then observers with $\mathfrak{F} = 0$ remain on Σ_0 given the initial condition $\xi(\lambda_0) = 0$. However, $K_{\alpha\beta} = 0$ places a restriction on the induced matter, namely $T^\alpha{}_\alpha = 0$. Finally, if observers are subject to a non-gravitational force such that $\mathfrak{F} = -K_{\alpha\beta}u^\alpha u^\beta$, then they can be confined to general Σ_0. The source of this centripetal confining force is not clear.

B. The thin braneworld scenario

We now move on to the widely-referenced thin braneworld scenarios proposed by Randall & Sundrum (RS). There are actually two different versions of the thin braneworld picture: the so-called RSI and RSII models. In both situations, one begins by assuming a 5-dimensional manifold with a non-zero cosmological constant Λ, which is often taken to be negative; i.e., the bulk is AdS$_5$. The Σ_0 hypersurface located at $\ell = 0$ represents a domain wall across which the normal derivative of the metric (the extrinsic curvature) is discontinuous. Those familiar with the thin-shell formalism in general relativity will realize that such a discontinuity implies that there is a thin 4-dimensional matter configuration living on Σ_0. The motivation for such a geometrical setup comes from the Horava & Witten theory mentioned in Sec. I, where standard model fields are effectively confined to a 3-brane in a higher-dimensional manifold. In the RS models, the distributional matter configuration corresponds to these matter fields. Now, if we stop here we have described the salient features of the RSII model. This scenario has drawn considerable interest in the literature, because for certain solutions the Kaluza-Klein spectrum of the graviton is such that Newton's $1/r^2$ law of gravitation is unchanged over astronomical length scales. By contrast, the RSI model differs from RSII by the addition of a second 3-brane located at some $\ell \neq 0$. The motivation

for the addition of the second brane comes from a possible solution of the hierarchy problem, which involves the disparity in size between the characteristic energies of quantum gravity and electroweak interactions. The idea is that the characteristic lengths, and hence energy scales, on the 3-branes are exponentially related by the intervening AdS_5 space. In what follows, we will concentrate mostly on the RSII scenario, although many of our comments can be applied to RSI.

When we apply the standard Israel junction conditions [46] to RSII, we find that the induced metric on the Σ_ℓ hypersurfaces must be continuous:

$$[h_{\alpha\beta}] = 0. \tag{29}$$

We adopt the common notation that $X^\pm \equiv \lim_{\ell\to 0^\pm} X$ and $[X] = X^+ - X^-$. In addition, the Einstein tensor of the bulk is given by

$$\hat{G}_{AB} = \Lambda g_{AB} + \kappa_5^2 T_{AB}^{(\Sigma)}, \quad T_{AB}^{(\Sigma)} = \delta(\ell)\, S_{\alpha\beta} e_A^\alpha e_B^\beta. \tag{30}$$

Here, the 4-tensor $S_{\alpha\beta}$ is defined as

$$[K_{\alpha\beta}] \equiv -\kappa_5^2 \varepsilon \left(S_{\alpha\beta} - \tfrac{1}{3} S h_{\alpha\beta} \right), \tag{31}$$

where $S = h^{\mu\nu} S_{\mu\nu}$. The interpretation is that $S_{\alpha\beta}$ is the stress-energy tensor of the standard models fields on the brane. To proceed further, we need to invoke another assumption of the RS model, namely the $\mathbb{Z}_2$ symmetry. This *ansatz* essentially states that the geometry on one side of the brane is the mirror image of the geometry on the other side. In practical terms, it implies

$$K_{\alpha\beta}^+ = -K_{\alpha\beta}^- \quad\Rightarrow\quad [K_{\alpha\beta}] = 2K_{\alpha\beta}^+, \tag{32}$$

which then gives

$$S_{\alpha\beta} = -2\varepsilon \kappa_5^{-2} P_{\alpha\beta}^+. \tag{33}$$

Therefore, the stress-energy tensor of conventional matter *on* the brane is entirely determined by the *extrinsic* curvature of Σ_0 evaluated in the $\ell \to 0^+$ limit.

The embedding problem takes on a slightly different flavor in the RS model. We still want to endow the Σ_0 3-brane with desirable properties, but we must also respect the $\mathbb{Z}_2$ symmetry and the discontinuous nature of the 5-geometry. We are helped by the fact that the constraint equations (19) are invariant under $K_{\alpha\beta} \to -K_{\alpha\beta}$. This suggests the following algorithm for the generation of a braneworld model:

1. Solve the constraint equations (19) on Σ_0 for $\Psi^a(y,0) = \Psi_0^+$ such that Σ_0 has the desired properties.

2. Obtain the solution for $\Psi^a(y,\ell)$ for $\ell > 0$ using the evolution equations $\partial_\ell \Psi^a$ and Ψ_0^+ as initial data.

3. Generate another solution of the constraint equations by making the switch $K_{\alpha\beta}(y,0) \to -K_{\alpha\beta}(y,0)$ in Ψ_0^+. Call the new solution Ψ_0^-.

4. Finally, derive the solution for $\Psi^a(y,\ell)$ for $\ell < 0$ using Ψ_0^- as initial data. The resulting solution for the bulk geometry will automatically be discontinuous and incorporate the $\mathbb{Z}_2$ symmetry about Σ_0.

This is of course very similar to the standard embedding procedure already outlined in Sec. II C, which allows us to apply the various conclusions of the Campbell-Magaard theorem to the thin braneworld scenario. In particular, we can still arbitrarily choose the induced metric on Σ_0 and have enough freedom to consistently solve the constraint equations. Therefore, *any solution of $(3 + 1)$-dimensional general relativity can be realized as a thin 3-brane in the RS scenario*. However, to accomplish this we lose control of the jump in extrinsic curvature $[K_{\alpha\beta}]$ across Σ_0, which is related to the stress-energy tensor of standard model fields living on the brane. So, if we fix the intrinsic geometry of the brane then the properties of conventional matter will be determined dynamically.

We can also consider the inverse of this problem. Instead of fixing $h_{\alpha\beta}(y,0)$, we can instead fix $S_{\alpha\beta}$. Then, equation (33) acts as $\frac{1}{2}n(n + 1) = 10$ additional field equations on Σ_0 for the elements of Ψ_0^+ or Ψ_0^-. By a similar argument as before, this means that we do not have enough residual freedom to completely choose $h_{\alpha\beta}(y,0)$ or $K_{\alpha\beta}^\pm$, which means that they are determined dynamically. This is a more traditional approach in that the configuration of conventional matter determines the induced metric on Σ_0 (albeit through unconventional field equations, as described below). It is interesting to note that the structure of the constraint equations allows one to either choose the geometry and solve for the matter, or choose the matter and solve for the geometry, just like Einstein's equations. This similarity means that a generic problem in general relativity also creeps into the braneworld scenario: the functional form of $S_{\alpha\beta}$ is not sufficient to determine the properties of the matter configuration — one also needs the metric. Since $h_{\alpha\beta}$ is determined by the stress-energy tensor, we cannot have *a priori* knowledge of the distribution of matter-energy. As in general relativity, the way out is to make some sort of *ansatz* for $h_{\alpha\beta}$ and $S_{\alpha\beta}$ and try to solve for the geometry and matter simultaneously [47].

The field equations on the brane are simply given by (22) with $K_{\alpha\beta}$ evaluated on either side of Σ_0. Usually, equation (33) is used to eliminate $K_{\alpha\beta}^\pm$, which yields the following expression for the Einstein 4-tensor on Σ_0:

$$G_{\alpha\beta} = \frac{\kappa_5^4}{12} \left[SS_{\alpha\beta} - 3S_{\alpha\mu}S^\mu{}_\beta + \left(\frac{3S^{\mu\nu}S_{\mu\nu} - S^2}{2} \right) h_{\alpha\beta} \right]$$
$$- \varepsilon E_{\alpha\beta} - \frac{1}{2}\lambda h_{\alpha\beta}. \tag{34}$$

Since this expression is based on the equations of constraint (19), it is entirely equivalent to the STM expression (27) when $\lambda = 0$. However, it is obvious that the two results are written in terms of different quantities. To further complicate matters, many workers write the braneworld field equations in terms of the non-unique

decomposition

$$S_{\alpha\beta} = \tau_{\alpha\beta} - \tilde{\lambda} h_{\alpha\beta}, \tag{35}$$

so that the final result is in terms of $\tau_{\alpha\beta}$ and $\tilde{\lambda}$ instead of $S_{\alpha\beta}$. On the other hand, STM field equations are often written in a non-covariant form, where partial derivatives of the induced metric with respect to ℓ appear explicitly instead of $K_{\alpha\beta}$ and $E_{\alpha\beta}$ [12, for example]. We believe that this disconnect in language is responsible for the fact that few workers have realized the substantial amount of overlap between the two theories; however, we should mention that the correspondence between "traditional" STM and brane world field equations has been previously verified in a special coordinate gauge by Ponce de Leon [48].

Let us now turn our attention to observers in the RSII scenario. To simplify matters, let us make the 5-dimensional gauge choice $\Phi = 1$ (our results will of course be independent of this choice). Then, the ℓ equation of motion (A6) for observers reduces to

$$\ddot{\ell} = \varepsilon(K_{\alpha\beta}u^\alpha u^\beta + \mathfrak{F}). \tag{36}$$

Now by using equation (31), we obtain

$$K_{\alpha\beta}^\pm u^\alpha u^\beta = \mp\tfrac{1}{2}\varepsilon\kappa_5^2 \left[S_{\alpha\beta}u^\alpha u^\beta - \tfrac{1}{3}(\kappa - \varepsilon\dot{\ell}^2)S \right]. \tag{37}$$

We can view this as the zeroth order term in a Taylor series expansion of $K_{\alpha\beta}u^\alpha u^\beta$ in powers of ℓ. In this spirit, the equation of motion can be rewritten as

$$\ddot{\ell} = -\tfrac{1}{2}\text{sgn}(\ell)\kappa_5^2 \left[S_{\alpha\beta}u^\alpha u^\beta - \tfrac{1}{3}(\kappa - \varepsilon\dot{\ell}^2)S \right] + \varepsilon\mathfrak{F} + O(\ell), \tag{38}$$

where

$$\text{sgn}(\ell) = \begin{cases} +1, & \ell > 0, \\ -1, & \ell < 0, \\ \text{undefined}, & \ell = 0, \end{cases} \tag{39}$$

and we remind the reader that $u^A u_A = u^\alpha u_\alpha + \varepsilon\xi^2 = \kappa$ (we will assume that u^A is timelike). From this formula, it is obvious that freely-falling observers ($\mathfrak{F} = 0$) can be confined to a small region around the brane if

$$S_{\alpha\beta}u^\alpha u^\beta - \tfrac{1}{3}(\kappa - \varepsilon\dot{\ell}^2)S > 0. \tag{40}$$

Of course, if the quantity on the left is zero or the coefficient of the $O(\ell)$ term in (38) is comparatively large, we need to look to the sign of the $O(\ell)$ term to decide if the particle is really confined. To get at the physical content of (40), let us make the slow-motion approximation $\dot{\ell}^2 \ll 1$. With this assumption, equation (40) can be rewritten as

$$\int dy \left\{ T_{AB}^{(\Sigma)} - \tfrac{1}{3}\text{Tr}[T^{(\Sigma)}]g_{AB} \right\} u^A u^B > 0. \tag{41}$$

This is an integrated version of the 5-dimensional strong energy condition as applied to the brane's stress-energy tensor, which includes a vacuum energy contribution from the brane's tension. Its appearance in this context is not particularly surprising; the Raychaudhuri equation asserts that matter that obeys the strong energy condition will gravitationally attract test particles. Therefore, we have shown that test observers can be gravitationally bound to a small region around Σ_0 if the total matter-energy distribution on the brane obeys the 5-dimensional strong energy condition, and their velocity in the ℓ-direction is small.

Finally, we would like to show that the equation of motion (38) has a sensible Newtonian limit. Let us demand that all components of the particle's spatial velocity satisfy $|u^i| \ll 1$ with $i = 1, 2, 3, 4$. Let us also neglect the brane's tension and assume that the density ρ of the confined matter is much larger than any of its principle pressures. Under these circumstances we have [47]:

$$S_{\alpha\beta}u^\alpha u^\beta \approx \rho, \quad h^{\alpha\beta}S_{\alpha\beta} \approx \kappa\rho. \tag{42}$$

The 5-dimensional coupling constant κ_5^2 is taken to be

$$\kappa_5^2 = \tfrac{3}{2}V_3 G_5, \tag{43}$$

where V_3 is the dimensionless volume of the unit 3-sphere and G_5 is the 5-dimensional Newton constant.[3] With these approximations, we get the following equation of motion for freely-falling observers:

$$\ddot{\ell} \approx -\tfrac{1}{2}\text{sgn}(\ell)V_3 G_5 \rho + O(\ell). \tag{44}$$

This is precisely the result that one would obtain from a Newtonian calculation of the gravitational field close to a 3-dimensional surface layer in a 4-dimensional space using Gauss's Law:

$$-\int_{\partial\mathcal{V}} \mathbf{g} \cdot d\mathbf{A} = V_3 G_5 \int_{\mathcal{V}} \rho\, d\mathcal{V}. \tag{45}$$

Here the integration 4-volume $\mathcal{V}$ is a small "pill-box" traversing the brane. Thus, we have shown that the full general-relativistic equation of motion in the vicinity of the brane (38) reduces to the 4-dimensional generalization of a known result from 3-dimensional Newtonian gravity in the appropriate limit.

---

[3] The gravity-matter coupling κ_5^2 is physically distinguished from Newton's constant G_5 in that the former is the coefficient of the stress-energy tensor in Einstein's equations while the latter is the constant that appears in the 5-dimensional generalization of Newton's law of universal gravitation; i.e., in the Newtonian limit of the 5-dimensional theory, the gravitation acceleration around a point mass is $G_5 M/r^3$. This expression for κ_5^2 given in equation (43) is consistent with the Newtonian force law and the (4+1)-version of Poisson's equation $\nabla_{(4)}^2\phi = V_3 G_5 \rho$, where $\nabla_{(4)}^2$ is the Laplacian operator in Euclidean 4-space. The fastest way to convince oneself of this is to compare the Newtonian and general relativistic expressions for the tidal acceleration between test particles, as in Section 4.3 of Wald [47].

In conclusion, we have seen that the Campbell-Magaard theorem says that it is possible to embed any solution of 4-dimensional general relativity in the RSII scenario. However, the price to be paid is that the matter content of the brane must then be determined dynamically. The field equations on the brane were seen to be similar to those of STM theory. We also found that test observers can be gravitationally confined to a small region around Σ_0 if the brane's stress-energy tensor obeys the 5-dimensional strong energy condition, and that their ℓ-equation of motion reduces to the Newtonian result in the appropriate limit. This implies that, as in the STM case, the requirement that observers be confined to Σ_0 imposes restrictions on the embedded spacetime.

C. The thick braneworld scenario

The last 5-dimensional model of our universe that we want to talk about is the so-called thick braneworld model [19, 20]. This scenario is essentially a "smoothed-out" version of the RSII picture, where the infinitely sharp domain wall at $\ell = 0$ is replaced with a continuously differentiable 4-dimensional geometric feature. There are two main motivations for the study of such an extension of RSII. First, since there is a natural minimum length scale in superstring/supergravity theories, the notion of an infinitely thin geometric defect must be viewed as an approximation. Second, one would like to see how these branes might arise dynamically from solutions of 5-dimensional supergravity theories, which are by necessity smooth solutions of some higher-dimensional action involving dilatonic scalar and other types of fields. The latter motivation means that the bulk may contain fields in addition to a non-vanishing vacuum energy in thick braneworld models. This means that our previous formulae cannot be applied to thick braneworlds generated by scalar fields or other higher-dimensional matter. For this reason, we will only consider thick braneworlds where such fields are absent. We mention in passing that we expect the results of this section can be straightforwardly generalized, largely because we know that the Campbell-Magaard theorem applies to situations where the higher-dimensional manifold is sourced by quite general stress-energy tensors [31, 32].

In moving from the RSII scenario to the thick braneworld, we retain the $\mathbb{Z}_2$ symmetry across the Σ_0 hypersurface. In order to satisfy the requirement that the extrinsic curvature be an odd function of ℓ, we need to have that Σ_0 has $K_{\alpha\beta} = 0$. This simplifies the situation enormously. The Einstein tensor on the brane then reduces to

$$G_{\alpha\beta} = -\varepsilon E_{\alpha\beta} - \tfrac{1}{2}\lambda h_{\alpha\beta}. \qquad (46)$$

In contrast to the $K_{\alpha\beta} = 0$ case for STM theory, the stress-energy of a thick brane embedded in a 5-dimensional Einstein space does not have vanishing trace unless $\lambda = 0$. Finally, we note that $K_{\alpha\beta} = 0$ allows us to

solve equation (A6) with $\ell = 0$: we can always construct freely-falling test-observer trajectories that lie within the brane. Since the extrinsic curvature of Σ_0 is fixed by the $\mathbb{Z}_2$ symmetry, the constraint equations (19) cannot be solved for arbitrary $h_{\alpha\beta}$. So the Campbell-Magaard theorem cannot be applied to this scenario.

Finally, we note that there is an additional requirement that must be met by viable thick braneworld models. Namely, the graviton ground state must be effectively localized near Σ_0. That is, 5-dimensional metric perturbations must satisfy a wave equation whose ground state is normalizable and peaked near the brane. The issue of normalizability clearly depends on the global properties of M, and is hence not addressable by the purely local formalism we have introduced thus far. At the risk of appearing pedantic, a short comment about this issue may be in order. Field equations like those of Einstein — or (46) above — take no notice of boundary conditions or other considerations such as topology; and the non-accessibility of such in the cosmological domain has led Wheeler (following Einstein) to argue that the global structure of 4-dimensions ought to be closed. It seems to us that a parallel argument can be made in the case of the thick braneworld scenario. We have no way of knowing the boundary conditions in 5-dimensions, so in the next section we turn our attention to a concrete but local example.

IV. AN EXAMPLE: A $\mathbb{Z}_2$ SYMMETRIC EMBEDDING OF RADIATION DOMINATED COSMOLOGY IN A 5-DIMENSIONAL VACUUM

In this, our penultimate section, we will consider a concrete application of the Campbell-Magaard theorem. Our aim is to take an empty solution of STM theory and interpret it in terms of the braneworld scenario. The induced metric on the brane will be seen to correspond to standard radiation-dominated cosmology. We will also investigate the behaviour of geodesics in the neighbourhood of the brane and the nature of the induced matter on hypersurfaces $\Sigma_\ell \neq \Sigma_0$.

We specialize to 5-dimensional manifolds with signature $(+ - - - -)$, which means that the normal to Σ_ℓ is spacelike, $\varepsilon = -1$, and timelike geodesics have $u^A u_A = +1$. Our starting point is a particularly interesting line element presented in ref. [41]. It is given by:

$$ds_{(5)}^2 = B^2(t,\ell)dt^2 - A^2(t,\ell)d\sigma_k^2 - d\ell^2, \qquad (47a)$$

$$A^2(t,\ell) = [\mu^2(t) + k]\ell^2 + 2\nu(t)\ell + \frac{\nu^2(t) + \mathcal{K}}{\mu^2(t) + k}, (47b)$$

$$B(t,\ell) = \frac{1}{\mu(t)}\frac{\partial}{\partial t}A(t,\ell). \qquad (47c)$$

Here $k = 0, \pm 1$ is the curvature index of the 3-geometry,

$d\sigma_k^2$ is the metric on 3-spaces of constant curvature

$$d\sigma_k^2 = d\chi^2 + S_k^2(\chi)(d\theta^2 + \sin^2\theta\, d\varphi^2), \quad (48a)$$

$$S_k(\eta) \equiv \begin{cases} \sin\eta, & k = +1, \\ \eta, & k = 0, \\ \sinh\eta, & k = -1, \end{cases} \quad (48b)$$

and $\mu(t)$ and $\nu(t)$ are arbitrary functions of time. The arbitrary constant $\mathcal{K}$ is related to the 5D Kretschmann scalar via

$$\hat{R}_{ABCD}\hat{R}^{ABCD} = \frac{72\mathcal{K}^2}{A^8(t,\ell)}. \quad (49)$$

This metric satisfies the 5-dimensional vacuum field equations

$$\hat{R}_{AB} = 0, \quad (50)$$

and is hence a solution of STM theory. This metric is of interest because the line element on Σ_ℓ hypersurfaces is isometric to standard Friedman-Lemaître-Robertson-Walker (FLRW) cosmologies, as we will see below.

We now want to use some of the functional arbitrariness in (47a) to obtain a thick braneworld model with $\mathbb{Z}_2$ symmetry about $\ell = 0$. Recalling that metrics with $\mathbb{Z}_2$ symmetry must have components that are even functions of ℓ, we see that we should set $\nu(t) = 0$ in equation (47a). If we also make the coordinate transformation $t \to \mu = \mu(t)$, we obtain the following form of the metric:

$$ds_{(5)}^2 = b^2(\mu,\ell)d\mu^2 - a^2(\mu,\ell)d\sigma_k^2 - d\ell^2, \quad (51a)$$

$$a^2(\mu,\ell) = (\mu^2 + k)\ell^2 + \frac{\mathcal{K}}{\mu^2 + k}, \quad (51b)$$

$$b(\mu,\ell) = \frac{[(\mu^2 + k)^2\ell^2 - \mathcal{K}]}{(\mu^2 + k)^{3/2}[(\mu^2 + k)^2\ell^2 + \mathcal{K}]^{1/2}}. \quad (51c)$$

This solution is manifestly $\mathbb{Z}_2$ symmetric about $\ell = 0$, implying $K_{\alpha\beta} = 0$ for the Σ_0 hypersurface. Notice that to ensure $a(\mu,\ell)$ is real-valued on the brane at $\ell = 0$, we need to demand

$$\frac{\mu^2 + k}{\mathcal{K}} > 0. \quad (52)$$

Our field equations (19) with $\lambda = 0$ predict $G^\alpha{}_\alpha = 0$ on the brane. This can be confirmed by direct calculation using the induced metric on Σ_0:

$$ds_{(\Sigma_0)}^2 = \frac{\mathcal{K}}{\mu^2 + k}\left[\frac{d\mu^2}{(\mu^2 + k)^2} - d\sigma_k^2\right], \quad (53)$$

which yields

$$G^\alpha{}_\beta\Big|_{\Sigma_0} = \frac{(\mu^2 + k)^2}{\mathcal{K}}\begin{pmatrix} +3 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}. \quad (54)$$

Here we have made a choice of 4-dimensional coordinates such that $e_\alpha^A = \delta_\alpha^A$. If $G_{\alpha\beta}$ is interpreted as the stress-energy tensor of a perfect fluid, it has a radiation-like equation of state $\rho = 3p$. Also note that the inequality (52) implies that the density and pressure are negative if $(\mu^2 + k) < 0$. Finally, if we change 4-dimensional coordinates via

$$\mu \to \eta = \eta(\mu) = \int_{\mu_0}^\mu \frac{dx}{x^2 + k}, \quad (55)$$

and carefully choose μ_0, we get the following line element on the brane:

$$ds_{(\Sigma_0)}^2 = \mathcal{K}S_k^2(\eta)[d\eta^2 - d\sigma_k^2]. \quad (56)$$

This is the standard solution for a radiation-dominated FLRW cosmology expressed in terms of the conformal time η [49]. We have thus obtained a $\mathbb{Z}_2$ symmetric embedding of a radiation dominated universe in a Ricci-flat 5-dimensional manifold.

Let us now consider the 5-dimensional geodesics of this model in the vicinity of the brane. The isotropy of the ordinary 3-space in the model means we can set $\dot{r} = \dot{\theta} = \dot{\varphi} = 0$ and deal exclusively with comoving trajectories. The Lagrangian governing such paths is

$$L = \tfrac{1}{2}\left[b^2(\mu,\ell)\dot{\mu}^2 - \dot{\ell}^2\right]. \quad (57)$$

We can obtain an equation for $\ddot{\ell}$ by extremizing the action, to yield:

$$\ddot{\ell} = -\frac{1}{2}\dot{\mu}^2\frac{\partial}{\partial\ell}b^2(\mu,\ell) = \left(\frac{3\dot{\mu}^2}{\mu^2 + k}\right)\ell + O(\ell^3). \quad (58)$$

We see that $\ell = 0$ is an acceptable solution of this equation, which is reasonable because Σ_0 has a vanishing extrinsic curvature. So, 5-dimensional geodesics can indeed be confined to the brane. What is more interesting is the behaviour of geodesics near the brane. The coefficient of ℓ on the righthand side of (58) is explicitly positive if the density and pressure of the matter in the embedded brane universe is positive, so test particles near the brane will experience a force pushing them away from $\ell = 0$. That is, the 3-brane in this model represents an unstable equilibrium for observers if the induced matter on Σ_0 has reasonable properties.

Finally, we note that we have been primarily concerned with the Σ_0 hypersurface. However, each of the hypersurfaces in the Σ_ℓ foliation can be interpreted as a different 4-dimensional universe. Since the hypersurfaces $\Sigma_\ell \neq \Sigma_0$ do not have $K_{\alpha\beta} = 0$, we expect that their induced matter does not have a radiation-like equation of state. To determine the properties of these universes, we use the induced metric on Σ_ℓ to calculate the Einstein 4-tensor,

which turns out to be given by:

$$G^{\alpha}{}_{\beta}\Big|_{\Sigma_{\ell}} = \kappa_4^2 \begin{pmatrix} +\rho & & & \\ & -p & & \\ & & -p & \\ & & & -p \end{pmatrix}, \qquad (59a)$$

$$\kappa_4^2 \rho(\mu,\ell) \equiv \frac{3(\mu^2+k)}{a^2(\mu,\ell)}, \qquad (59b)$$

$$\kappa_4^2 p(\mu,\ell) \equiv \frac{2a(\mu,\ell)+(\mu^2+k)b(\mu,\ell)}{a^2(\mu,\ell)b(\mu,\ell)}. \qquad (59c)$$

From these expressions for the density and pressure of the induced matter on Σ_{ℓ}, we can derive the following expression for the so called quintessence parameter:

$$\gamma(\mu,\ell) = \frac{p(\mu,\ell)}{\rho(\mu,\ell)} = \frac{1}{3}\left[\frac{\mathcal{K}+3\ell^2(\mu^2+k)^2}{\mathcal{K}-\ell^2(\mu^2+k)^2}\right]. \qquad (60)$$

For $\ell = 0$, we recover our previous result $\gamma = 1/3$ for all μ. For $\ell \neq 0$, we obtain $\gamma \to -1$ as $\mu \to \infty$. Hence, the universes located at $\ell \neq 0$ approach the vacuum-dominated FLRW solution (i.e., $\rho = -p$) for late times. These results are very plausible from the physical perspective.

What we have done immediately above may be summarized with a view to future work. We have used the arbitrariness in a known solution of STM theory to generate a $\mathbb{Z}_2$ symmetric embedding of FLRW radiation-dominated cosmologies in a 5-dimensional Ricci-flat manifold. The Σ_0 hypersurface is an example of how a 3-brane with zero extrinsic curvature has $G^{\alpha}{}_{\alpha} = 0$ when $\lambda = 0$. We have also presented a concrete realization of the confinement of 5-dimensional test observers to a 3-brane. However, the equilibrium position of observers on Σ_0 was shown to be unstable if the density and pressure of the induced matter on the brane is positive. The 4-dimensional spacetimes corresponding to the Σ_{ℓ} hypersurfaces other than Σ_0 were seen to approach the de Sitter FLRW universe for late times. These results are intriguing, but preliminary. We note that the components of g_{AB} in this model are non-separable functions of μ and ℓ, so a complete analysis of the tensor and scalar waves admitted by this model will be non-trivial.

V. SUMMARY

In this paper we have derived field equations (19) for three spin-2 fields $\{h_{\alpha\beta}, K_{\alpha\beta}, E_{\alpha\beta}\}$ living on an n-dimensional hypersurface Σ_0 embedded in an $(n+1)$-dimensional Einstein space M. We have used these equations to give a heuristic proof of a generalized Campbell-Magaard theorem, which states that it is possible to embed any n-dimensional psuedo-Riemannian manifold in an $(n+1)$-dimensional space with or without a cosmological constant. We also demonstrated that instead of embedding Σ_0 in M with an arbitrary metric $h_{\alpha\beta}$, we can instead embed it with arbitrary extrinsic curvature $K_{\alpha\beta}$.

These results were then applied to three different 5-dimensional models of the universe. In STM theory, we found that the theorem allowed us to realize any solution of general relativity as a 4-surface in a 5-dimensional vacuum spacetime. However, by examining a (4+1)-splitting of the equation of motion of test particles, we found that observers will not be confined to Σ_0 unless it has vanishing extrinsic curvature. If the latter condition is imposed, then the induced matter on Σ_0 has a radiation-like equation of state. In the RSII thin braneworld scenario, we found that one could also embed any solution of 4-dimensional relativity on a 3-brane. However, in so doing one loses control of the stress-energy tensor of standard model fields living on Σ_0. We showed that test observers in this model can be confined to a small region around Σ_0 if the total brane stress-energy tensor obeys the 5-dimensional strong energy condition. Finally, we found that the $\mathbb{Z}_2$ symmetry in the thick braneworld model *requires* that $K_{\alpha\beta} = 0$ on the brane. This implies a restrictive 4-geometry, and we cannot embed arbitrary spacetimes if the bulk contains only vacuum energy. Also, test observers are naturally confined to the brane in this scenario.

We considered for illustrative purposes a class of solutions from STM theory. This class included enough arbitrariness for us to impose the $\mathbb{Z}_2$ symmetry about the $\ell = 0$ hypersurface. We found that the induced matter on the brane had a radiation-like equation of state, in agreement with previous results. Using 4-dimensional coordinate transformations, we demonstrated that the geometry on Σ_0 exactly matched the standard radiation-dominated FLRW models. We also found that $\ell = 0$ was an unstable equilibrium for test observers if the density and pressure in those models is positive. We also looked at the properties of the universes on $\ell \neq 0$ hypersurfaces and found that they are asymptotically the same as de Sitter FLRW cosmologies.

One of the main themes that has emerged from our considerations is the mathematical similarity between the STM and braneworld scenarios. Despite the fact that they have very different motivations, we have found that the field equations of each theory are definitely related. The thin braneworld expression for $G_{\alpha\beta}$ (34) reduces to the STM formula (27) in the $\lambda \to 0$ limit and with an appropriate choice of gauge. The $K_{\alpha\beta} = 0$ case of STM theory matches the $\lambda = 0$ case (46) in the thick braneworld scenario we considered. Also, the thin braneworld field equations reduce to the thick braneworld case when $K_{\alpha\beta} = 0$. The reason for these correspondences, which have been noted before [48], is seen to be something rather simple: If we wish to embed our 4-dimensional matter-filled world in an empty 5-dimensional universe, the constraint equations (19) have to be obeyed. A technical corollary of this is that the numerous known solutions of STM theory [12] can be reinterpreted as brane models.

Acknowledgments

APPENDIX A: COVARIANT SPLITTING OF TEST PARTICLE EQUATIONS OF MOTION

In ref. [40], a covariant $(4 + 1)$-splitting of the 5-dimensional geodesic equation was performed for the case $n^A n_A = -1$. This resulted in two equations of motion, one for motion parallel to Σ_ℓ and another for motion orthogonal to Σ_ℓ. We propose to generalize those results to the current situation where $n^A n_A = \varepsilon$, and to include the possibility that observers' trajectories are subject to the influence of some non-gravitational force per unit mass, denoted by f^A. The relevant 5D equations of motion are given by:

$$u^B \nabla_B u^A = f^A, \quad \text{(A1a)}$$
$$u_A u^A = \kappa \equiv +1, 0, -1, \quad \text{(A1b)}$$
$$u^A \equiv dx^A/d\lambda \equiv \dot{x}^A. \quad \text{(A1c)}$$

Here λ is the 5-dimensional affine parameter, and an overdot indicates $d/d\lambda = u^A \partial_A$. Generalizing the calculations found in ref. [40], the 5-dimensional equations of motion can be written as

$$u^\beta_{;\alpha} u^\alpha = -\varepsilon \xi (K^{\alpha\beta} u_\alpha + e_B^\beta n^A \nabla_A u^B) + f^\beta, \quad \text{(A2a)}$$
$$\dot{\xi} = K_{\alpha\beta} u^\alpha u^\beta + \varepsilon \xi n^A u^B \nabla_A n_B + \mathfrak{F}, \quad \text{(A2b)}$$
$$\kappa = h_{\alpha\beta} u^\alpha u^\beta + \varepsilon \xi^2. \quad \text{(A2c)}$$

These involve the definitions

$$u^\alpha \equiv e_A^\alpha u^A, \quad \xi \equiv n_A u^A, \quad \text{(A3a)}$$
$$f^\alpha \equiv e_A^\alpha f^A, \quad \mathfrak{F} \equiv n_A f^A. \quad \text{(A3b)}$$

It was also demonstrated in ref. [40] that

$$u^\alpha = \dot{y}^\alpha + N^\alpha \dot{\ell}, \quad \xi = \varepsilon \Phi \dot{\ell}, \quad \text{(A4)}$$

where Φ and N^α are the lapse and shift introduced in Section II A. Consider the last relation and the the identities

$$(\ln \Phi)_{;\beta} = -\varepsilon e_\beta^B n^A \nabla_A n_B, \quad \text{(A5a)}$$
$$\dot{\Phi} = u^\alpha \Phi_{;\alpha} + \varepsilon \xi n^A \nabla_A \Phi. \quad \text{(A5b)}$$

Then equation (A2b) can be rewritten in the mixed but instructive form

$$\ddot{\ell} = \frac{\varepsilon}{\Phi}(K_{\alpha\beta} u^\alpha u^\beta + \mathfrak{F}) - \dot{\ell}\left[2u^\beta(\ln \Phi)_{;\beta} + \dot{\ell} n^A \nabla_A \Phi\right]. \quad \text{(A6)}$$

In this paper, we will be primarily concerned with this expression, which governs motion perpendicular to Σ_ℓ.

We conclude by saying a few words about the κ parameter: Our formalism for dealing with 5-dimensional geodesics can be applied to timelike, spacelike or null paths. For each of these cases, a choice of metric signature must be made before κ can be specified. For example, if the 5-dimensional metric signature is $(+ - - - \pm)$, then timelike paths have $u^A u_A = +1$ or $\kappa = +1$.

[1] T. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **K1**, 966 (1921).
[2] O. Klein, Z. Phys. **37**, 895 (1926).
[3] D. W. Joseph, Phys. Rev. **126**, 319 (1962).
[4] K. Akama, Lect. Notes Phys. **176**, 267 (1982), hep-th/0001113.
[5] V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. **B125**, 136 (1983).
[6] M. Visser, Phys. Lett. **B159**, 22 (1985), hep-th/9910093.
[7] G. W. Gibbons and D. L. Wiltshire, Nucl. Phys. **B287**, 717 (1987), hep-th/0109093.
[8] I. Antoniadis, Phys. Lett. **B246**, 377 (1990).
[9] P. S. Wesson and J. Ponce de Leon, J. Math. Phys. **33**, 3883 (1992).
[10] P. S. Wesson et al., Int. J. Mod. Phys. **A11**, 3247 (1996).
[11] J. M. Overduin and P. S. Wesson, Phys. Rept. **283**, 303 (1997), gr-qc/9805018.
[12] P. S. Wesson, *Space-Time-Matter* (World Scientific, Singapore, 1999).
[13] P. Horava and E. Witten, Nucl. Phys. **B475**, 94 (1996), hep-th/9603142.
[14] N. Arkani-Hamed, S. Dimopoulos, and G. R. Dvali, Phys. Lett. **B429**, 263 (1998), hep-ph/9803315.
[15] N. Arkani-Hamed, S. Dimopoulos, and G. R. Dvali, Phys. Rev. **D59**, 086004 (1999), hep-ph/9807344.
[16] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. R. Dvali, Phys. Lett. **B436**, 257 (1998), hep-ph/9804398.
[17] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999), hep-ph/9905221.
[18] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999), hep-th/9906064.
[19] O. DeWolfe, D. Z. Freedman, S. S. Gubser, and A. Karch, Phys. Rev. **D62**, 046008 (2000), hep-th/9909134.
[20] C. Csaki, J. Erlich, T. J. Hollowood, and Y. Shirman, Nucl. Phys. **B581**, 309 (2000), hep-th/0001033.
[21] M. Pavsic and V. Tapia (2000), gr-qc/0010045.
[22] B. Riemann, Abh. Königl. gesellsch. **13**, 1 (1868).
[23] L. Schläfli, Ann. di Mat. 2e série **5**, 170 (1871).
[24] M. Janet, Ann. Soc. Polon. Math. **5**, 38 (1926).

[25] E. Cartan, Ann. Soc. Polon. Math. **6**, 1 (1927).

[26] C. Burstin, Rec. Math. Moscou (Math. Sbornik) **38**, 74 (1931).

[27] J. Campbell, *A Course on Differential Geometry* (Claredon, Oxford, 1926).

[28] L. Magaard, Ph.D. thesis, Kiel (1963).

[29] E. Anderson and J. E. Lidsey, Class. Quant. Grav. **18**, 4831 (2001), gr-qc/0106090.

[30] F. Dahia and C. Romero (2001), gr-qc/0109076.

[31] E. Anderson, F. Dahia, J. E. Lidsey, and C. Romero (2001), gr-qc/0111094.

[32] F. Dahia and C. Romero (2001), gr-qc/0111058.

[33] J. Nash, Ann. Math. **63**, 20 (1956).

[34] C. J. Clarke, Proc. Roy. Soc. London **A314**, 417 (1970).

[35] R. E. Greene, *Isometric embedding of Riemannian and pseudo-Riemannian manifolds*, no. 97 in Memoirs Am. Math. Soc. (American Mathematical Society, Providence, RI, 1970).

[36] C. Romero, R. K. Tavakol, and R. Zalaletdinov, Gen. Rel. Grav. **28**, 365 (1996).

[37] M. D. Maia (1995), gr-qc/9512002.

[38] J. E. Lidsey, R. K. Tavakol, and C. Romero, Mod. Phys. Lett. **A12**, 2319 (1997).

[39] J. E. Lidsey, C. Romero, R. K. Tavakol, and S. Rippl, Class. Quant. Grav. **14**, 865 (1997), gr-qc/9907040.

[40] S. S. Seahra, Phys. Rev. **D65**, 124004 (2002), gr-qc/0204032.

[41] H. Liu and P. S. Wesson, Astrophysical Journal **562**, 1 (2001), gr-qc/0107093.

[42] T. Shiromizu, K.-i. Maeda, and M. Sasaki, Phys. Rev. **D62**, 024012 (2000), gr-qc/9910076.

[43] J. Ponce de Leon, Gen. Rel. Grav. **20**, 539 (1988).

[44] S. S. Seahra and P. S. Wesson, Class. Quant. Grav. **19**, 1139 (2002), gr-qc/0202010.

[45] H. Ishihara, Phys. Rev. Lett. **86**, 381 (2001), gr-qc/0007070.

[46] W. Israel, Nuovo Cim. **B44S10**, 1 (1966).

[47] R. M. Wald, *General Relativity* (Univ. of Chicago, Chicago, 1984).

[48] J. Ponce de Leon, Mod. Phys. Lett. **A16**, 2291 (2001), gr-qc/0111011.

[49] J. A. Peacock, *Cosmological Physics* (Cambridge University Press, Cambridge, 1999).